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Pseudo Edge Geodetic Number and Perfect Edge Geodetic Graph

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Abstract- An edge geodetic set of G is a set $S \subseteq V(G)$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The edge geodetic number $g_1(G)$ of G is the minimum order of its edge geodetic sets and any edge geodetic set of order $g_1(G)$ is an edge geodetic basis of G or a g_1 – set of G. A set of vertices S' in G is called Pseudo edge geodetic set if the set of vertices which are not belongs to any edge geodetic basis of G and. The Maximum cardinality of a Pseudo edge geodetic set of G is its Pseudo edge geodetic number and is denoted by g_1' .A Pseudo edge geodetic set of size $g_1'(G)$ is said to be a g_1' set. Pseudo edge geodetic number of certain classes of graphs is determined. Also if $g_1'(G)$ =0 then G is called Perfect edge geodetic graph.

Keywords: Edge geodetic number, Pseudo edge geodetic number, Perfect edge geodetic graph
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1.INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [3,10]. The distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u,v)$ is called an u-v geodesic. A vertex x is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. The minimum eccentricity among the vertices of G is the *radius*, *rad* G or $r(G)$ and the maximum eccentricity is its *diameter*, *diam* G of G. A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a *geodetic basis*. The geodetic number of a graph was introduced in [4] and further studied in [6, 12,13]. An *edge geodetic set* of G is a set $S \subseteq V$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The *edge geodetic number* $g_1(G)$ of G is the minimum order of its edge geodetic sets and any edge geodetic set of order $g_1(G)$ is an *edge geodetic basis* of G or a g_1-set of G. The edge geodetic number of a graph G is studied in [1,11, 14, 15]. $N(v) = \{ u \in V(G) : uv \in E(G)\}$ is called the neighborhood of the vertex v in G. For any set S of vertices of G, the *induced subgraph* $<S>$ is the maximal subgraph of G with vertex set S. A vertex v is a *simplicial vertex* of a graph G if $<N(v)>$ is complete. A simplex of a graph G is a subgraph of G which is a complete graph. If $e = \{u, v\}$ is an edge of a graph G with $d(u) = 1$ and $d(v) > 1$, then we call e a pendent edge, u a leaf and v a support vertex. Let $L(G)$ be the set of all leaves of a graph G.

It is easily seen that a Pseudo edge geodetic set is not in general a complementary edge geodetic set in a graph G. Also the converse is not valid in general. This has motivated us to study the new Pseudo edge geodetic conception of a graph. We investigate those subsets of vertices of a graph that are not belongs to edge geodetic set. We call these sets Pseudo edge geodetic sets. We call the maximum cardinality of the edge geodetic dominating set of G, the Pseudo edge geodetic number of G.

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2. Pseudo Edge Geodetic Number

Definition2.1: Let G=(V,E) be a connected graph and S be the edge geodetic basis of G. Then the set of vertices which are not belongs to any edge geodetic basis of G is the Pseudo edge geodetic set S′ of G and the maximum cardinality of S′ is called Pseudo edge geodetic number and is denoted by g'_1

Example 2.2 For the graph G given in Figure 2.1, $S = \{v_1, v_2, v_4\}$ is the edge geodetic basis for G

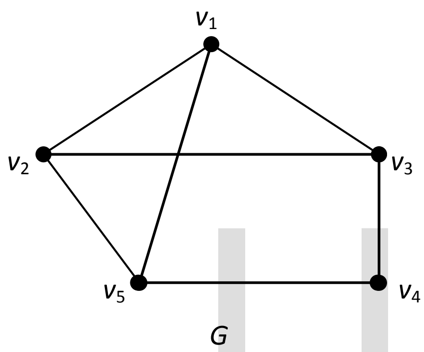

Figure 2.1

The Pseudo edge geodetic set $S' = \{v_3, v_5\}$ and hence $g'_1(G)=2$.

Remark 2.3 for the graph G given in Figure 2.1, $S' = S^c$ and hence $g'_1(G) = |V-S|$. But in general S′ is not the complement set of S. For the graph G given in Figure 2.2, $S = \{v_1, v_3, v_4, v_6\}$ and $S_1 = \{v_1, v_2, v_4, v_5\}$ are two edge geodetic bases and so $g'_1(G) = 0$.

Theorem2.4 Let $S_1, S_2, S_3 \ldots\ldots S_n$ are the edge geodetic bases of G then $g'_1(G) = |\cap_1^n S_i^c|$

Proof: Let S′ be the Pseudo edge geodetic set of G .It is enough to prove that $S' = \cap_1^n S_i^c$. Let $v \in V$ such that $v \in S'$ .Then v doesn't belongs to any edge geodetic basis of G. Hence $v \notin S_i \ \forall i (1 \leq i \leq n)$

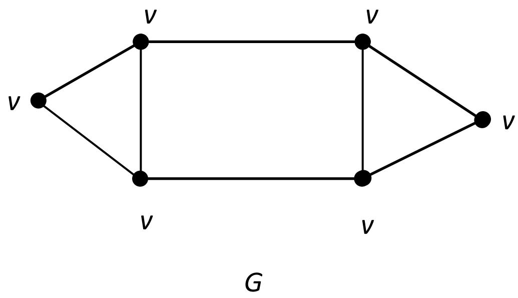

Figure 2.2

$v \in S_i^c \forall i (1 \leq i \leq n)$

$v \in \cap_1^n S_i^c$

$S' \subseteq \cap_1^n S_i^c$

Let u be a vertex of G such that $u \in \cap_1^n S_i^c$

$u \in S_i^c \forall i (1 \leq i \leq n)$

$u \notin S_i \ \forall i (1 \leq i \leq n)$

Hence $u \in S'$ and so $S' = \cap_1^n S_i^c$

Theorem2.5 Let G=(p,q) be a connected graph ,then $g'_1(G)+g_1(G) = p$ if and only if $f_1(G)=0$

Proof: Suppose $g'_1(G)+g_1(G) = p$. Let S′ be the Pseudo edge geodetic set and S be the edge geodetic basis of G. Then S′+S=p and hence S′=p-S. Then by the Theorem2.4 S is the unique edge geodetic basis of G and So forcing edge geodetic number $f_1 (G) = 0$ .Conversely suppose $f_1(G)=0$, then G has unique edge geodetic basis S. By the Theorem2.4 $S' = S^c$ and hence $g'_1(G)+g_1(G) = p$.

Theorem 2.6 For any connected G $0 \leq g'_1(G) \leq p-2$

Proof: An edge geodetic set needs at least two vertices and therefore $g'_1(G) \leq p-2$. Clearly the set of all vertices of K_p is the edge geodetic basis of G so that g'1(G) =0 Thus $0 \leq g'_1(G) \leq p\text{-}2$.

Remark 2.7 The bounds in Theorem 2.6 are sharp. For the complete graph $K_p (p \geq 2)$, $g'_1(K_p) = 0$. The set of non end-vertices of a path P_p $(p \geq 2)$ is its Pseudo edge geodetic set so that $g'_1(P_p) = p\text{-}2$. Thus the path P_p has the largest possible Pseudo edge geodetic number p-2 and that the complete graph has the least Pseudo edge geodetic number 0.

Theorem2.8 No extreme vertex, in particular no end vertex belongs to Pseudo edge geodetic set.

Proof. Let S be an edge geodetic set of G and v be an extreme vertex of G. Let $\{v_1, v_2, \ldots, v_k\}$ be the neighbors of v and vv_i $(1 \leq i \leq k)$ be the edges incident on v. Since v is an extreme vertex, v_i and v_j are adjacent for $i \neq j$ $(1\leq i, j \leq k)$ so that any geodesic which contains vv_i $(1 \leq i \leq k)$ is either v_iv or $u_1u_2, \ldots, u_lv_iv$, where each u_i $(1 \leq i \leq l)$ is different from v_i. Hence it follows that $v \in S$ and hence v∉ S'

Theorem2.9 For any connected graph G, every cut-vertex of G belongs to Pseudo edge geodetic set of G.

Proof. Let S,S' be any edge geodetic basis, Pseudo edge geodetic set of G and let $v \in S$ be any vertex. We claim that v is not a cut-vertex of G. Suppose that v is a cut-vertex of G. Let $G_1, G_2, \ldots, G_r$ $(r \geq 2)$ be the components of $G - v$. Then v is adjacent to at least one vertex of G_i for every $i(1 \leq i \leq r)$. Let $S_1 = S - \{v\}$. Let uw be an edge of G which lies on a geodesic P joining a pair of vertices say x and v of S. Assume without loss of generality that $x \in G_1$. Since v is adjacent to at least one vertex of each G_i $(1 \leq i \leq r)$, assume that v is adjacent to a vertex y in G_k $(k \neq 1)$. Since S is an edge geodetic set, vy lies on a geodesic Q joining v and a vertex z of S such that z (possibly y itself) must necessarily belong to G_k. Thus $z \neq v$. Now, since v is a cut-vertex of G, the union $P \cup Q$ of the two geodesics P and Q is obviously a geodesic in G joining x and z in S and thus the edge uw lies on this geodesic joining the two vertices x and z of S_1. Thus we have proved that every edge that lies on a geodesic joining a pair of vertices x and v of S also lies on a geodesic joining two vertices of S_1 . Hence it follows that every edge of G lies on a geodesic joining two vertices of S_1, which shows that $S_1$is an edge geodetic set of G. Since $|S_1| = |S| - 1$, this contradicts the fact that S is an edge geodetic basis of G. Hence $v \notin S$ so that $v \in$S'

Corollary 2.10. For any connected graph G with k cut vertices, $k \leq g'_1(G) \leq p\text{-}2$

Proof. This follows from Theorems 2.6 and 2.9.

Corollary 2.11. For any non-trivial tree T, the Pseudo edge geodetic number $g'_1(T)$ equals the number of non end-vertices in T. In fact, the set of all non end-vertices of T is the Pseudo edge geodetic set of T.

Proof. This follows from Theorems 2.8 and 2.9.

Corollary 2.12 For the complete graph K_p $(p \geq 2)$, $g'_1(K_p) = 0$

Proof. Since every vertex of the complete graph K_p $(p \geq 2)$ is an extreme vertex, by the Theorem2.8 $g'_1(K_p) = 0$

Theorem 2.13. If G has exactly one vertex v of degree $p-1$, then $g'_1(G) = 1$

Proof: If G has exactly one vertex v of degree $p-1$, then $g_1(G) = p-1$ and G has a unique edge geodetic basis consisting of all the vertices of G other than v. Then by the Theorem2.5 $g'_1(G) = 1$

Theorem 2.14. For the complete bipartite graph $G = K_{m,n}$,

(i) $g'_1(G) = 1$ if $n \geq 2$, $m = 1$.

(ii) $g'_1(G) =$ max $\{m, n\}$ if $m, n \geq 2$.

(iii) $g'_1(G) = 0$ if $m = n$

Proof. (i) This follows from Corollary 2.13.

(ii) Let $m, n \geq 2$. First assume that $m < n$. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of G. Let $S = U$. We prove that S is an edge geodetic basis of G. Any edge u_iw_j ($1 \leq i \leq m$, $1 \leq j \leq n$) lies on the geodesic $u_iw_ju_k$ for any $k \neq i$ so that S is an edge geodetic set of G. Let T be any set of vertices such that $|T| < |S|$. If $T \subsetneq U$, then there exists a vertex $u_i \in U$ such that $u_i \notin T$. Then for any edge u_iw_j ($1 \leq j \leq n$), the only geodesics containing u_iw_j are $u_iw_ju_k$ ($k \neq i$) and $w_ju_iw_l$ ($l \neq j$) and so u_iw_j cannot lie in a geodesic joining two vertices of T. Thus T is not an edge geodetic set of G. If $T \subsetneq W$, again T is not an edge geodetic set of G by a similar argument. If $T \subsetneq U \cup W$ such that T contains at least one vertex from each of U and W, then, since $|T| < |S|$, there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin T$ and $w_j \notin T$. Then clearly the edge u_iw_j does not lie on a geodesic connecting two vertices of T so that T is not an edge geodetic set. Thus in any case T is not an edge geodetic set of G. Hence S is an edge geodetic basis so that $g_1(K_{m,n}) = |S| = m$. Now, if $m = n$, we can prove similarly that $S = U$ or W is an edge geodetic basis of G. Thus G has unique edge geodetic basis U. Then by Theorem 2.5 S′=W and hence $g'_1(G)$ =n= max {m,n}.

(iii) Suppose m= n. then as in (ii) U=$\{u_1, u_2, \ldots, u_m\}$,W =$\{w_1, w_2, \ldots, w_m\}$ are the only edge geodetic bases of G.Hence by the Theorem 2.4, $g'_1(G) = 0$.

3. Perfect Edge Geodetic Graph

Definition 3.1 A connected graph G is said to be perfect edge geodetic graph if every vertex *of* G lies in anyone of the edge geodetic basis of G .That is, G is perfect edge geodetic graph if $g'_1(G)=0$

Example 3.2 For the graph G given in Figure 2.2, $S = \{v_1, v_3, v_4, v_6\}$ and $S_1 = \{v_1, v_2, v_4, v_5\}$ are two edge geodetic bases and so $g'_1(G) = 0$.Hence G is a perfect edge geodetic graph.

Theorem3.3 If G has more than one vertex of degree p-1 then G is perfect edge geodetic graph.

Proof: If all the vertices are of degree $p-1$, then $G = K_p$ the by the **Corollary 2.12** $g'_1(G) = 0$ and hence G is perfect edge geodetic graph. Otherwise, let $v_1, v_2, \ldots, v_k$($2 \leq k \leq p-2$) be the vertices of degree $p-1$. Suppose $g'_1(G)>0$ then $g_1(G) < p$. Let S be an edge geodetic basis of G such that $|S| < p$. Then S contains all the vertices $v_1, v_2, \ldots, v_k$. Let v be a vertex such that v

$\notin S$. Then deg (v) < p – 1. Since any two of v_1, v_2,….,v_k are adjacent, the edge vv_i ($1 \leq i \leq k$) cannot lie on a geodesic joining a pair of vertices v_j and v_l ($j \neq l$). Similarly, since any v_j is adjacent to any vertex of S, which is different from v_1, v_2, …, v_k, the edge vv_i($1 \leq i \leq k$) cannot lie on a geodesic joining a vertex v_j and a vertex of S, which is different from v_1, v_2, …, v_k. Now, let u and w be vertices of S different from v_1, v_2, …, v_k. Since v_i is adjacent to both u and w and $d(u, v) \leq 2$, the edge vv_i cannot lie on a geodesic joining u and w. Thus we see that the edges vv_i ($1 \leq i \leq k$) do not lie on any geodesic joining a pair of vertices of S, which is a contradiction to the fact that S is an edge geodetic basis of G. Hence $g'_1(G) = 0$

Theorem 3.4 any complete bipartite graph G=K_{mn} is perfect if and only if m=n

Proof: Suppose G=K_{mn} is perfect .Then by definition $g'_1(G) = 0$. Suppose m< n. Then by the Theorem 2.14 (ii) $g'_1(G)$ = n, is a contradiction to the hypothesis .Hence m=n. Conversely suppose m=n, then by the Theorem 2.14 (iii) $g'_1(G) = 0$, then G=K_{mn} is perfect.

Theorem 3.5 any complete graph K_p ($p \geq 2$) is perfect

Proof: The proof follows **Corollary 2.12**

Theorem 3.6 Any even cycle C_p is perfect edge geodetic graph.

Proof: Since C_p is an even cycle $g_1(G)=2$,also $S_1=\{v_1, v_{\frac{p}{2}+1}\}$,$S_2=\{v_2, v_{\frac{p}{2}+2}\}$..................$S_{\frac{p}{2}}=\{v_{\frac{p}{2}}, v_p\}$ are the only edge geodetic basis of C_p. Then by the Theorem 2.4 $g'_1(G) = 0$and hence C_p is perfect edge geodetic graph

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